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On a Generalization of a Theorem of Nash-Williams

BERT FASSBENDER

Mathematisches Institut der Universität zu Köln, Weyertal 86-90, D-5000 Cologne, Germany

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Let G be a simple k-connected graph of order $v \geqslant 3$ with minimum degree δ and independence number α. Fraisse generalized a theorem of Nash-Williams by proving that G is hamiltonian if $\delta \geqslant \max\{\alpha + k - 2, (1/(k+1))(v + k(k-1))\}$. We give an alternative proof of this theorem. © 1991 Academic Press, Inc.

We use the terminology of Bondy and Murty [4] and consider only simple graphs. Our starting point is the following well-known result due to Nash-Williams [8].

THEOREM 1. *Let G be a 2-connected graph of order v with minimum degree δ and independence number α. If $\delta \geqslant \max\{\alpha, \frac{1}{3}(v+2)\}$, then G is hamiltonian.*

A cycle C of a graph G is called a D_λ-cycle if every component of $G - C$ has order less than λ. Using a deep result on D_λ-cycles, Fraisse [7] obtained the following generalization of Theorem 1.

THEOREM 2. *Let G be a k-connected graph of order $v \geqslant 3$ with minimum degree δ and independence number α. If $\delta \geqslant \max\{\alpha + k - 2, (1/(k+1))(v + k(k-1))\}$, then G is hamiltonian.*

The purpose of the present note is to give an independent proof based on well-known results on longest cycles. Following Bondy, we say that a nontrivial connected graph is *m-path-connected* if any two of its vertices are joined by a path of length at least m; the trivial graph is said to be 0-path-connected. Bondy proved the following in [1] (see also [2]):

THEOREM 3. *Let G be a k-connected graph of order $v \geqslant 3$ such that the degree-sum of any $k+1$ independent vertices is at least $v + k(k-1)$, and let*

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C be a longest cycle in G. Then $G-C$ contains no $(k-1)$-path-connected subgraph.

This result and the following two theorems are used in the proof of Theorem 2.

THEOREM 4 [6]. *A 2-connected graph with minimum degree at least m is m-path-connected.*

THEOREM 5 [3, 5]. *Let G be a 2-connected graph on at least four vertices, and let u, v, and w be vertices of G with $u \neq v$. If each vertex in $V(G)\setminus\{u, v, w\}$ has degree at least m, then G contains a (u, v)-path of length at least m.*

If H is a subgraph and v a vertex of the graph G, we denote by $N_H(v)$ the set of all G-neighbors of v belonging to H and by $d_H(v)$ the cardinality of $N_H(v)$. If v belongs to H, then $\alpha_H(v)$ denotes the maximum cardinality of an independent vertex set in H containing v.

PROOF OF THEOREM 2

Suppose that G satisfies the hypothesis of the theorem but is not hamiltonian. Let $C = c_1 c_2, ..., c_r c_1$ be a longest cycle in G, and let H be a component of $G-C$ and v a vertex of H. Consider the set

$$S := \{c_{i+1} \mid c_i \in N_C(v)\},$$

where the indices of the c's are to be understood modulo r.

Since C is a longest cycle in G, the set $S \cup \{x\}$ is independent in G for any vertex x of H, hence $\alpha \geqslant \alpha_H(v) + d_C(v)$. On the other hand, $\alpha + k - 2 \leqslant \delta \leqslant d_C(v) + d_H(v)$ by hypothesis. Therefore,

$$d_H(v) \geqslant \alpha_H(v) + k - 2, \tag{1}$$

which implies

$$d_H(v) \geqslant k-1. \tag{2}$$

We now show that H contains a $(k-1)$-path-connected subgraph.

If $k \leqslant 2$, this is obvious by (2), and so we may assume that $k \geqslant 3$. If H is 2-connected, then H itself is $(k-1)$-path-connected by (2) and Theorem 4. Otherwise, let B be a block of H that contains exactly one cut vertex w of H, and let $v \in V(B)\setminus\{w\}$. Since v is not adjacent to any vertex of $H-B$, we have $\alpha_H(v) \geqslant 2$, hence

$$d_H(v) \geqslant k \tag{3}$$

by (1), and consequently $v(B) \geqslant k+1$. Because $k \geqslant 3$, the block B contains at least four vertices. Using (3) and Theorem 5, we infer that B is a k-path-connected subgraph of H. By hypothesis, the degree-sum of any $k+1$ vertices is at least $v + k(k-1)$; thus we have arrived at a contradiction to Theorem 3.

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